THE SUPPORT THEOREM FOR THE COMPLEX RADON TRANSFORM OF DISTRIBUTIONS

A.B. SEKERIN

Abstract. The complex Radon transform $\hat{F}$ of a rapidly decreasing distribution $F \in \mathscr{O}'_C(\mathbb{C}^n)$ is considered. A compact set $K \subset \mathbb{C}^n$ is called linearly convex if the set $\mathbb{C}^n \setminus K$ is a union of complex hyperplanes. Let $\hat{K}$ denote the set of complex hyperplanes which meet K. The main result of the paper establishes the conditions on a linearly convex compact K under which the support theorem for the complex Radon transform is true: from the relation $\mathrm{supp}(\hat{F}) \subset \hat{K}$ it follows that $F \in \mathscr{O}'_C(\mathbb{C}^n)$ is compactly supported and $\mathrm{supp}(F) \subset K$.

If f is the function defined on $\mathbb{R}^n$ ($\mathbb{C}^n$), the classical real (complex) Radon transform Rf of f is the function defined on hyperplanes; the value of Rf at a given hyperplane is the integral of f over that hyperplane. For the theory of the Radon transform we refer to J. Radon [10], F. John [6], [7], I.M.Gel'fand, M.I.Graev, and N.Ya. Vilenkin [1], S. Helgason [2], [3], D. Ludwig [8], A. Hertle [4]. One of the basic results on the classical Radon transform is Helgason's support theorem [2]: A rapidly decreasing function must vanish outside a ball if its real Radon transform does. This theorem holds for every convex compact set in $\mathbb{R}^n$ and remains valid for rapidly decreasing distributions [4].

In the present paper we prove the support theorem for the complex Radon transform of distributions.

Notations. For $z, w \in \mathbb{C}^n$ we write $\langle z, w\rangle = \sum z_j w_j$. $B^n(z,R) := \{w \in \mathbb{C}^n \mid |w - z| < R\}$ denotes the euclidean ball of center z and radius r in $\mathbb{C}^n$. If X is a set, we denote by $\bar{X}$ the closure of X. The standard Lebesgue measure in $\mathbb{C}^n$ is $d\omega_{2n}$. S^{2n-1} denotes the unit sphere in $\mathbb{C}^n$, and $d\sigma$ is the area element on S^{2n-1}. For n-tuples $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ of non-negative integers, we denote by $\partial^p \bar{\partial}^q$ the partial derivative

$$\frac{\partial^{|p|+|q|}}{\partial z_1^{p_1} \ldots \partial z_n^{p_n} \partial \bar{z}_1^{q_1} \ldots \partial \bar{z}_n^{q_n}}$$

of order $|p| + |q| = p_1 + \ldots + p_n + q_1 + \ldots + q_n$. Similarly, for $z = (z_1, \ldots, z_n)$ we write $z^p = z_1^{p_1} \ldots z_n^{p_n}$, $\bar{z}^q = \bar{z}_1^{p_1} \ldots \bar{z}_n^{q_n}$. For a domain $\Omega \subset \mathbb{C}^n$, we denote by $\mathscr{S}(\Omega)$, $\mathscr{D}(\Omega)$, and $\mathscr{E}(\Omega)$ the spaces of rapidly decreasing C^∞ functions, C^∞ functions with compact support, and C^∞ functions, respectively. The dual spaces $\mathscr{S}'(\Omega)$, $\mathscr{D}'(\Omega)$, and $\mathscr{E}'(\Omega)$ are the spaces of tempered distributions, distributions, and distributions with compact support, respectively.

2000 *Mathematics Subject Classification.* Primary 44A12, Secondary 46F10, 46F12, 30E99.

Key words and phrases. The Radon transform, complex variables, spaces of distributions.

The author gratefully acknowledges support of the Deutscher Akademischer Austauschdienst (DAAD) for the visit at Mathematisches Institut der Heinrich-Heine-Universität Düsseldorf, where this research was carried out.

If $\varphi \in \mathscr{S}(\mathbb{C}^n)$, the standard complex Radon transform of φ (denoted by $\hat{\varphi}$) is defined by

$$\hat{\varphi}(\xi, s) = \frac{1}{|\xi|^2} \int\limits_{\langle z,\xi\rangle = s} \varphi(z)\, d\lambda(z), \tag{1}$$

where $(\xi, s) \in (\mathbb{C}^n \backslash 0) \times \mathbb{C}$, and $d\lambda(z)$ is the area element on the hyperplane $\{z : \langle z, \xi\rangle = s\}$. For a set $A \subset \mathbb{C}^n$, we denote by $\hat{A}$ the set of all $(\xi, s) \in (\mathbb{C}^n \backslash 0) \times \mathbb{C}$ such that the hyperplane $\{z : \langle z, \xi\rangle = s\}$ meets A. A set $A \subset \mathbb{C}^n$ is called linearly convex if, for every $w \notin A$, there is a complex hyperplane $\{z : \langle z, \xi\rangle = s\}$ which contains w and does not meet A (see Martineau [9]).

We use the approach of Gel'fand et al. [1] to introduce the complex Radon transform of distributions. Let $X = S^{2n-1} \times \mathbb{C}$, and let $\mathscr{E}(X)$ be the set of complex-valued functions $\varphi(w, s)$ on $S^{2n-1} \times \mathbb{C}$ which satisfy the following conditions:

(a) Functions $\varphi(w, s)$ are infinitely differentiable with respect to s.
(b) For all $p, q \geq 0$ the derivatives

$$\frac{\partial^{p+q}\varphi(w, s)}{\partial s^p \partial \bar{s}^q}$$

are continuous on $S^{2n-1} \times \mathbb{C}$.
(c) $\varphi(we^{i\theta}, se^{i\theta}) = \varphi(w, s)$ for all $\theta \in [0, 2\pi]$.

We give $\mathscr{E}(X)$ the topology defined by the system of seminorms

$$q_k(f) = \max_{k_1+k_2 \leq k} \max_{|s| \leq k} \max_{w \in S^{2n-1}} \left| \frac{\partial^{k_1+k_2} f(w, s)}{\partial s^{k_1} \partial \bar{s}^{k_2}} \right|.$$

By $\mathscr{D}(X)$ we denote the space of all compactly supported functions in $\mathscr{E}(X)$. We give $\mathscr{D}(X)$ the standard topology of the inductive limit of the spaces

$$\mathscr{D}_m = \left\{ \varphi \in \mathscr{E}(X) : \mathrm{supp}(\varphi) \subset S^{2n-1} \times \{|s| \leq m\} \right\}.$$

Let $R\mathscr{D}(X)$ be the subspace of $\mathscr{D}(X)$ formed by the Radon transforms $\hat{\varphi}$ of functions in $\mathscr{D}(\mathbb{C}^n)$ (the equality $\hat{\varphi}(we^{i\theta}, se^{i\theta}) \equiv \hat{\varphi}(we^{i\theta}, se^{i\theta})$ follows for $\varphi \in \mathscr{D}(\mathbb{C}^n)$ from the definition of $\hat{\varphi}$). Similarly, we define the subspace $R\mathscr{S}(X)$ of $\mathscr{S}(X)$.

The dual Radon transform is the operator $R^* : \mathscr{E}(X) \to \mathscr{E}(\mathbb{C}^n)$ given by

$$[R^*(f)](z) = \int\limits_{S^{2n-1}} f(w, \langle z, w\rangle)\, d\sigma(w).$$

It is easy to see that the operator R^* is continuous. It follows from the definition of the Radon transform that

$$\int\limits_{\mathbb{C}^n} [R^*(f)](z)\varphi(z)\, d\omega_{2n}(z) = \int\limits_{\mathbb{C}} \int\limits_{S^{2n-1}} f(w, s)\hat{\varphi}(w, s)\, d\sigma(w) d\omega_2(s) \tag{2}$$

for every function $\varphi \in \mathscr{D}(\mathbb{C}^n)$.

Let $M_{\mathscr{D}}$ be the subspace of $\mathscr{D}(X)$ formed by the functions

$$\psi(w, s) = \frac{\partial^{2n-2}\hat{\varphi}(w, s)}{\partial s^{n-1} \partial \bar{s}^{n-1}}, \quad \hat{\varphi} \in R\mathscr{D}(X). \tag{3}$$

We give $M_{\mathscr{D}}$ the topology induced from $\mathscr{D}(X)$.

Definition 1. *Let $F \in \mathscr{D}'$. The Radon transform RF of F is the functional on $M_{\mathscr{D}}$ given by*

$$\langle RF, \psi \rangle = \langle F, R^* \psi \rangle. \tag{4}$$

For every function $\varphi \in \mathscr{S}(\mathbb{C}^n)$ the following inversion formula holds [1, p. 118]:

$$\varphi(z) = (-1)^{n-1} c_n R^* \left(\frac{\partial^{2n-2} \hat{\varphi}(w, s)}{\partial s^{n-1} \partial \bar{s}^{n-1}} \right), \tag{5}$$

where $\hat{\varphi}(w, s)$ is the Radon transform of φ, and $c_n > 0$. It follows from the inversion formula (5) that for each function $\psi \in M_{\mathscr{D}}$ the function $R^*(\psi)(z)$ belongs to $\mathscr{D}(\mathbb{C}^n)$. Therefore the functional RF is well defined.

Definition 2. *We say that the Radon transform RF of a distribution $F \in \mathscr{D}'$ is defined as a distribution if the functional RF given by (4) can be extended to a continuous functional on $\mathscr{D}(X)$.*

It has been shown in [4] that there are distributions in $\mathbb{R}^m$ whose real Radon transforms are not defined as distributions. It is natural to suppose that there are such examples in the case of the complex Radon transform. If the distribution F is given by the function $f(z) \in \mathscr{S}(\mathbb{C}^n)$, then it follows from (5) and (2) that the Radon transform RF is defined as a distribution and it is given by the function $\hat{f}(w, s)$.

We denote by $\mathscr{O}'_C(\mathbb{C}^n)$ the space of rapidly decreasing distributions [5, p. 419]. A distribution $T \in \mathscr{D}'(\mathbb{C}^n)$ belongs to $\mathscr{O}'_C(\mathbb{C}^n)$ if and only if for every $k \in \mathbb{Z}$ the distribution $(1 + |x|^2)^k T$ is integrable; i.e.,

$$(1 + |x|^2)^k T = \sum_{|p|+|q| \le m(k)} \partial^p \bar{\partial}^q \mu_{pq}(k), \tag{6}$$

where $m(k) \in \mathbb{N}$ and $\{\mu_{pq}\}(k)$ is a finite family of bounded measures on $\mathbb{C}^n$. In particular, every distribution with compact support is rapidly decreasing.

Let $T \in \mathscr{O}'_C(\mathbb{C}^n)$. We show that equality (4) defines the extension of the Radon transform RT to a continuous linear functional on $\mathscr{D}(X)$. Let $h(w, s) \in \mathscr{D}(X)$ be such that $|h(w, s)| \le 1$. There is $R > 0$ such that $h(w, s) = 0$ for $|s| \ge R$, and we have

$$|[R^* h](z)| \le \int_{S^{2n-1}} |h(w, \langle z, w \rangle)| d\sigma(w) \le \int_{|\langle z, w \rangle| \le R} d\sigma(w) \le d_n \max\left(1, \frac{R^2}{|z|^2}\right), \tag{7}$$

where $d_n > 0$. Suppose that the sequence $\{h_N(w, s)\}$ in $\mathscr{D}(X)$ converges to 0. Then, for every multi-indices p and q, we have

$$\partial^p \bar{\partial}^q \left[R^*(h_N) \right](z) = \int_{S^{2n-1}} \frac{\partial^{|p|+|q|}}{\partial s^{|p|} \partial \bar{s}^{|q|}} h_N(w, \langle z, w \rangle) w^p \bar{w}^q d\sigma(w). \tag{8}$$

There exists $R > 0$ such that $\operatorname{supp}(h_N) \subset S^{2n-1} \times \{s : |s| \le R\}$ for all N. Then it follows from (7) and (8) that

$$\left| \partial^p \bar{\partial}^q \left[R^*(h_N) \right](z) \right| \le d_n \max\left(1, \frac{R^2}{|z|^2}\right) \max_{w,s} \left| \frac{\partial^{|p|+|q|}}{\partial s^{|p|} \partial \bar{s}^{|q|}} h_N(w, s) \right|. \tag{9}$$

This means that the functions $\left[R^*(h_N)\right](z)$, together with derivatives of all orders, vanish at infinity. By the definition of the topology of $\mathscr{D}(X)$ we have

$$\lim_{N\to\infty}\max_{w,s}\left|\frac{\partial^{|p|+|q|}}{\partial s^{|p|}\partial\bar{s}^{|q|}}h_N(w,s)\right|=0. \tag{10}$$

We set $k=0$ in (6). Then we obtain from (6) and (4) that

$$\langle RT,h_N\rangle=\langle T,[R^*h_N]\rangle=\sum_{|p|+|q|\le m}(-1)^{|p|+|q|}\int\limits_{\mathbb{C}^n}\partial^p\bar{\partial}^q[R^*h_N](z)d\mu_{pq}(z).$$

Since the measures μ_{pq} are bounded, it follows from (9) and (10) that $\langle RT,h_N\rangle\to 0$ as $N\to\infty$. Thus, for every $T\in\mathscr{O}'_C(\mathbb{C}^n)$, the functional RT is well-defined and continuous on $\mathscr{D}(X)$.

Theorem 1. *Let $T\in\mathscr{O}'_C(\mathbb{C}^n)$ and let $K\subset\mathbb{C}^n$ be a linearly convex compact set. Suppose that for every $z\notin K$ there exists a hyperplane $P=\{\lambda:\langle\lambda,w_0\rangle=s_0\}$ satisfying the following conditions:*
(i) P contains z.
(ii) P does not meet K.
(iii) The set $\mathbb{C}\setminus K_{w_0}$ is connected, where $K_{w_0}=\{\langle\lambda,w_0\rangle\}_{\lambda\in K}$ is the projection of K on w_0. Then T has support in K if and only if its Radon transform RT has support in $\hat{K}$.

Remark. Theorem 1 was proved by the author in the special case in which the distribution T is given by a compactly supported continuous function [12]. The proof of Theorem 1 is based on the properties of the convolution of T and smooth compactly supported functions. As in the proof of the similar theorem for the real Radon transform and convex compact sets [4], the proof of Theorem 1 can be easily reduced to the case of regular distributions if for small enough $\varepsilon>0$ the set

$$K_\varepsilon=\bigcup_{z\in K}\bar{B}^n(z,\varepsilon)$$

also satisfies the conditions (i)-(iii). It should be noted that, in contrast to the case of convex compacts, there are examples of compact sets K satisfying (i)-(iii) such that the set K_ε does not satisfy the condition (iii) for every $\varepsilon>0$. Since it has been shown in [12] that assumption (iii) in Theorem 1 is essential, Theorem 1 is not a simple consequence of the result of [12].

Proof of Theorem 1. Suppose that $T\in\mathscr{O}'_C(\mathbb{C}^n)$ has support in K. Then $T\in\mathscr{E}(\mathbb{C}^n)$. Let $h(w,s)\in\mathscr{D}(X)$ be such that $\operatorname{supp}(h)\subset X\setminus\hat{K}$. If $z\in K$, then the point $(w,\langle z,w\rangle)$ belongs to $\hat{K}$ for every $w\in S^{2n-1}$. Therefore the functions

$$[R^*h](z)=\int\limits_{S^{2n-1}}h(w,\langle z,w\rangle)d\sigma(w),$$

$$\partial^p\bar{\partial}^q\left[R^*(h)\right](z)=\int\limits_{S^{2n-1}}\frac{\partial^{|p|+|q|}}{\partial s^{|p|}\partial\bar{s}^{|q|}}h(w,\langle z,w\rangle)w^p\bar{w}^q d\sigma(w)$$

vanish on K. So $[R^*h](z)$ is an infinitely differentiable function which, together with derivatives of all orders, vanishes on the support of the distribution T. Then we have

$\langle T, R^*h\rangle = 0$. Thus, for each $h \in \mathscr{D}(X)$ with $\operatorname{supp}(h) \in X \setminus \hat{K}$ we have $\langle RT, h\rangle = \langle T, [R^*h]\rangle = 0$. This means that $\operatorname{supp}(RT) \subset \hat{K}$.

Before proving the second statement of Theorem 1, we have to show that the dual Radon transform and the convolution operation commute:

Lemma 1. *Let $\varphi(z) \in \mathscr{D}(\mathbb{C}^n)$. Then for every $\psi(w,s) \in \mathscr{E}(X)$ the following formula holds:*

$$\varphi * [R^*\psi] = R^*[\hat{\varphi} *_s \psi],$$

where $\hat{\varphi}(w,s)$ is the Radon transform of φ, and $*_s$ denotes the convolution with respect to the second variable s.

Proof. For every function $\alpha(z) \in \mathscr{D}(\mathbb{C}^n)$ we have

$$\int\limits_{\mathbb{C}^n} (\varphi * [R^*\psi])(z)\alpha(z)d\omega_{2n}(z) = \int\limits_{\mathbb{C}^n} [R^*\psi](z)\left(\alpha * \varphi_1\right)(z)d\omega_{2n}(z), \tag{11}$$

where $\varphi_1(z) = \varphi(-z)$. Let J be the integral on the right-hand side of (11). It follows from (2) that

$$J = \int\limits_{S^{2n-1}\times\mathbb{C}} \psi(w,s)\widehat{\alpha * \varphi_1}(w,s)d\sigma(w)d\omega_2(s),$$

where $\widehat{\alpha * \varphi_1}(w,s)$ is the Radon transform of the convolution $\alpha * \varphi$. We have [1, p.p. 116-117]

$$\widehat{\alpha * \varphi_1}(w,s) = (\hat{\alpha} *_s \hat{\varphi}_1)(w,s), \quad \hat{\varphi}_1(w,s) = \hat{\varphi}(-w,s) = \hat{\varphi}(w,-s).$$

Then

$$J = \int\limits_{S^{2n-1}\times\mathbb{C}} \psi(w,s)(\hat{\alpha} *_s \hat{\varphi}_1)(w,s)d\sigma(w)d\omega_2(s) = \int\limits_{S^{2n-1}\times\mathbb{C}} (\psi *_s \hat{\varphi})(w,s)\hat{\alpha}(w,s)d\sigma(w)d\omega_2(s).$$

In view of (2), we have

$$J = \int\limits_{\mathbb{C}^n} R^*[\varphi *_s \psi](z)\alpha(z)d\omega_{2n}(z).$$

Then it follows from (11) that

$$\int\limits_{\mathbb{C}^n} \left\{(\varphi * [R^*\psi])(z) - R^*[\varphi *_s \psi](z)\right\}\alpha(z)d\omega_{2n}(z) = 0$$

for every $\alpha(z) \in \mathscr{D}(\mathbb{C}^n)$. Therefore $(\varphi * [R^*\psi])(z) \equiv R^*[\varphi *_s \psi](z)$. The lemma is proved.

Now suppose that the support of the Radon transform RT of a distribution $T \in \mathscr{O}'_C(\mathbb{C}^n)$ is contained in $\hat{K}$. Let $\{\alpha_m(z)\}_{m=1}^{\infty}$ be a sequence of smooth functions on $\mathbb{C}^n$ with $\operatorname{supp}(\alpha_m) \subset \{z : |z| \leq 1/m\}$ that converges in the space of measures to the delta function at the origin. We assume that the functions $\alpha_m(z)$ are even, i.e., $\alpha_m(-z) = \alpha_m(z)$. We set $T_m = T * \alpha_m$. Then the function $T_m(z)$ belongs to $\mathscr{S}(\mathbb{C}^n)$ [11, p. 244], and $T_m \to T$ in $\mathscr{O}'_C(\mathbb{C}^n)$ [4]. Denote by K_m the compact set

$$K_m = \bigcup_{z\in K} \bar{B}^n(z, 1/m).$$

Let $\hat{T}_m(w,s)$ be the Radon transform of $T_m(z)$. We show that $\operatorname{supp}(\hat{T}_m) \subset \hat{K}_m$. The hyperplane $\{z : \langle z, w\rangle = s\}$ meets K_m if and only if there are $z' \in K$, $z'' \in \bar{B}^n(0,1/m)$ such that $\langle z', w\rangle = s - \langle z'', w\rangle$. Therefore

$$\hat{K}_m = \bigcup_{(w,s)\in\hat{K}} \left(\{w\} \times \bar{B}^1(s,1/m)\right). \tag{12}$$

Let $h(w,s) \in \mathscr{D}(S^{2n-1} \times \mathbb{C})$ be such that $\operatorname{supp}(h) \cap \hat{K}_m = \emptyset$. Since the functions α_m are even, it follows from (4) that

$$\langle RT_m, h\rangle = \langle T_m, R^*(h)\rangle = \langle T * \alpha_m, R^*(h)\rangle = \langle T, \alpha_m * R^*(h)\rangle.$$

Then by Lemma 1, we have $\langle T, \alpha_m * R^*(h)\rangle = \langle T, R^*(\hat{\alpha}_m *_s h)\rangle$. Then

$$\langle RT_m, h\rangle = \langle T, R^*(\hat{\alpha}_m *_s h)\rangle = \langle RT, \hat{\alpha}_m *_s h\rangle. \tag{13}$$

We claim that $\hat{K} \cap \operatorname{supp}(\hat{\alpha}_m *_s h) = \emptyset$. Indeed, suppose that $(w_0, s_0) \in \hat{K} \cap \operatorname{supp}(\hat{\alpha}_m *_s h)$. This implies (since $\hat{\alpha}_m(w,s) = 0$ for $|s| \geq 1/m$) that for some $s_1 \in \bar{B}^1(0,1/m)$ we have $(w_0, s_0 + s_1) \in \operatorname{supp}(h)$. By (12) we also have $(w_0, s_0 + s_1) \in \hat{K}_m$, which contradicts that $\operatorname{supp}(h) \cap \hat{K}_m = \emptyset$. Therefore $\hat{K} \cap \operatorname{supp}(\hat{\alpha}_m *_s h) = \emptyset$, and it follows from (13) (since $\operatorname{supp}(RT) \subset \hat{K}$) that $\langle RT_m, h\rangle = 0$. Therefore

$$\operatorname{supp}(RT_m) \subset \hat{K}_m. \tag{14}$$

As remarked above, the functions $T_m(z)$ belong to $\mathscr{S}(\mathbb{C}^n)$. Then the distributions RT_m are given by the Radon transforms $\hat{T}_m(w,s)$ of functions $T_m(z)$.

In view of (12), there exist $R > 0$ such that for all m the sets $\hat{K}_m$ are contained in the set $\{(w,s) : |s| \leq R\}$. Let $R_{\mathbb{R}}T_m(w,t)$ be the real Radon transform of $T_m(z)$, that is

$$R_{\mathbb{R}}T_m(w,t) = \int_{\operatorname{Re}\langle z,\bar{w}\rangle = t} T_m(z)\,d\lambda(z),$$

where $d\lambda(z)$ is the area element on the real hyperplane $\{z : \operatorname{Re}\langle z, \bar{w}\rangle = t\}$. Then we have

$$R_{\mathbb{R}}T_m(w,t) = \int_{-\infty}^{\infty} \hat{T}_m(\bar{w}, t + ix)\,dx.$$

Since $\hat{K}_m \subset \{(w,s) : |s| \leq R\}$, it follows from (14) that $R_{\mathbb{R}}T_m(w,t) = 0$ for $|t| \geq R$. Then by the Helgason's support theorem, the supports of the functions $T_m(z)$ are compact.

To complete the proof of Theorem 1, we need the following lemma:

Lemma 2. *Under the hypotheses and notation of Theorem 1, there exist, for every $z_0 \notin K$, a neighborhood V_{z_0} and $\delta > 0$ such that the functions $T_m(z)$ vanish on V_{z_0} for $m \geq 1/\delta$.*

Proof. Fix $z_0 \notin K$. Then there exists a point $(w_0, s_0) \in S^{2n-1} \times \mathbb{C}$ such that $\{z : \langle z, w_0\rangle = s_0\} \cap K = \emptyset$, $\langle z_0, w_0\rangle = s_0$ and the set $\mathbb{C} \setminus \{\langle z, w_0\rangle\}_{z\in K}$ is connected. Then $(w_0, \langle z_0, w_0\rangle) \notin \hat{K}$. We set

$$A = \left\{s \in \mathbb{C} \mid (w_0, s) \in \hat{K}\right\}, \quad A_m = \left\{s \in \mathbb{C} \mid (w_0, s) \in \hat{K}_m\right\}.$$

It follows from (12) that

$$A_m = \bigcup_{s\in A} \bar{B}^1(s, 1/m).$$

By definition of $\hat{K}$, for every $s \in A$ there exists $z \in K$ such that $\langle z, w_0\rangle = s$. Then $A = \{\langle z, w_0\rangle\}_{z\in K}$. Similarly $A_m = \{\langle z, w_0\rangle\}_{z\in K_m}$. Since the sets K and K_m are compact, it follows that the sets A and A_m are also compact. For some $R > 0$ we have $A \cup A_m \subset \bar{B}^1(0, R)$. Since $\langle z_0, w_0\rangle \notin A$, there is $\gamma > 0$ such that $\langle z_0 + \lambda, w_0\rangle \notin A$ for every $\lambda \in \bar{B}^n(0, \gamma)$. Hence the convex compact set $\Gamma_1 = \left\{\langle z, w_0\rangle, z \in \bar{B}^n(z_0, \gamma)\right\}$ and the set A do not intersect. Fix $s_1 \in \{s \in \mathbb{C} : |s| > R\}$. Then $s_1 \in \mathbb{C} \setminus A$. Since the set $\mathbb{C} \setminus A$ is connected, there exists a broken line $\Gamma_2 \subset \mathbb{C} \setminus A$ joining s_1 to the point $\langle z_0, w_0\rangle$. Thus $(\Gamma_1 \cup \Gamma_2) \cap A = \emptyset$. Then, since the sets $\Gamma_1 \cup \Gamma_2$ and A are compact, there exists $\delta \in (0, 1)$ such that for all $m \geq 1/\delta$ we have

$$\left\{(\Gamma_1 \cup \Gamma_2) + B^1(0, \delta)\right\} \cap \left\{A + \bar{B}^1(0, 1/m)\right\} = \emptyset,$$

that is $\{(\Gamma_1 \cup \Gamma_2) + B^1(0, \delta)\} \cap A_m = \emptyset$. Put

$$D = \{s \in \mathbb{C} : |s| > R\} \cup \left\{(\Gamma_1 \cup \Gamma_2) + B^1(0, \delta)\right\}.$$

By construction D is a connected unbounded open set containing the point $\langle z_0+\lambda, w_0\rangle$ for every $\lambda \in \bar{B}^n(0, \gamma)$. We have by the definition of the sets A_m that $(D\times\{w_0\})\cap\hat{K}_m = \emptyset$ for $m \geq 1/\delta$. Then it follows from (14) that $(D \times \{w_0\}) \cap \operatorname{supp}(\hat{T}_m) = \emptyset$ for $m \geq 1/\delta$. Since the supports of T_m are compact, it follows from [12, Thm. 2] that for every $\lambda \in \bar{B}^n(0, \gamma)$ and $m \geq 1/\delta$ the functions $T_m(z)$ vanish on the hyperplane $\{z : \langle z, w_0\rangle = \langle z_0 + \lambda, w_0\rangle\}$. Then, for every $z \in \bar{B}^n(z_0, \gamma)$ and $m \geq 1/\delta$, we have $T_m(z) = 0$. The lemma is proved.

As mentioned above, $T_m \to T$ in $\mathscr{O}'_C(\mathbb{C}^n)$. This means that

$$\lim_{m\to\infty} \langle T_m, \varphi\rangle = \langle T, \varphi\rangle, \quad \forall\varphi \in \mathscr{O}_C(\mathbb{C}^n), \tag{15}$$

where $\mathscr{O}_C(\mathbb{C}^n)$ is the space of all infinitely differentiable functions f on $\mathbb{C}^n$ for which there exist an integer k such that $(1 + |x|^2)^k\partial^p\bar{\partial}^q f(z)$ vanishes at infinity for all p, q [5, p. 173]. Since $\mathscr{D}(\mathbb{C}^n) \subset \mathscr{O}_C(\mathbb{C}^n)$, formula (15) holds for every $\varphi \in \mathscr{D}(\mathbb{C}^n)$. Let $\varphi \in \mathscr{D}(\mathbb{C}^n)$ be such that $\operatorname{supp}(\varphi) \cap K = \emptyset$. By Lemma 2 for every $z \in \operatorname{supp}\varphi$ there are $\delta(z) > 0$ and a ball $B^n(z, \gamma(z))$ such that $T_m(z) = 0$ on $B^n(z, \gamma(z))$ for $m \geq 1/\delta(z)$. Since the support of φ is compact, it can be covered by a finite union of balls $B^n(z_k, \gamma(z_k))$, where $k = 1, 2 \ldots, N$. Setting $\delta_0 = \min\{\delta(z_k), 1 \leq k \leq N\}$, we have $T_m(z) = 0$ for $z \in \operatorname{supp}(\varphi)$ and $m \geq 1/\delta_0$. Then it follows from (15) that

$$\langle T, \varphi\rangle = \lim_{m\to\infty} \langle T_m, \varphi\rangle = 0.$$

Since $\varphi \in \mathscr{D}(\mathbb{C}^n)$ is an arbitrary function such that $\operatorname{supp}(\varphi) \cap K = \emptyset$, we have $\operatorname{supp}(T) \subset K$. The theorem is proved.

References

[1] I.M.Gel'fand, M.I.Graev, and N.Ya. Vilenkin, *Generalized functions. Volume. 5: Integral geometry and representation theory*, Academic Press, New York and London, 1966.

[2] S. Helgason, The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds, *Acta Math.*, **113** (1965), 153-180.

[3] S. Helgason, *The Radon transform*, Birkhäuser, Boston, 1980.

[4] A. Hertle, Continuity of the Radon transform and its inverse on Euclidean space, *Math. Z.*, **184** (1983), 165-192.

[5] J. Horvath, *Topological vector spaces and distributions*, vol. 1, Addison-Wesley, Reading, Massachusetts, 1966.

[6] F. John, Bestimmung einer Funktion aus ihren Integralen über gewisse Manningfaltigkeiten, *Math. Ann.*, **109** (1934), 488-520.

[7] F. John, *Plane waves and spherical means*, Interscience Publishers, New York, 1955.

[8] D. Ludwig, The Radon transform on Euclidean space, *Comm. Pure Appl. Math.*, **19** (1966), 49-81.

[9] A. Martineau, Sur la notion d′ensemble fortement lineellement convexe, *Ann. Acad. Brasil. Ciênc.*, **40** (1968), 427-435.

[10] J. Radon, Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Manningfaltigkeiten, *Ber. Verh. Sächs. Akad.*, **69** (1917), 262-277.

[11] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.

[12] A. B. Sekerin, A support theorem for the complex Radon transform, *Collect. Math.*, **50** (1999), 221-227.

The Orel State University, Komsomolskaya 95, Orel, 302026, Russia
E-mail address: `sekerin@orel.ru`